

The Aharonov–Bohm effect in scattering theory

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Abstract

The Aharonov–Bohm effect is considered as a scattering event with nonrelativistic charged particles of the wavelength which is less than the transverse size of an impenetrable magnetic vortex. The quasiclassical WKB method is shown to be efficient in solving this scattering problem. We find that the scattering cross section consists of two terms, one describing the classical phenomenon of elastic reflection and another one describing the quantum phenomenon of diffraction; the Aharonov–Bohm effect is manifested as a fringe shift in the diffraction pattern. Both the classical and the quantum phenomena are independent of the choice of a boundary condition at the vortex edge, providing that probability is conserved. We show that a propagation of charged particles can be controlled by altering the flux of a magnetic vortex placed on their way.

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1 Introduction

A comprehensive description of electromagnetism in classical theory is given in terms of the electromagnetic field strength acting locally and directly on charged matter. This is not the case in quantum theory where a state of charged matter is influenced by electromagnetic field even in the situation when the region of the nonvanishing field strength does not overlap with the region accessible to charged matter; the indispensable condition is that the latter region be non-simply-connected. In particular, a magnetic field of an infinitely long current-carrying solenoid is made impenetrable to charged matter; such a field configuration may be denoted as an impenetrable magnetic vortex. Perhaps, W. Ehrenberg and R. E. Siday were the first to discover theoretically the purely quantum effect which is due to the influence of the vector potential of the impenetrable magnetic vortex on the motion of a charged particle, and they proposed an interference experiment to detect this [1]. Ten years later the effect was rediscovered by Y. Aharonov and D. Bohm who made an important addition by predicting also an effect which is due to the influence of the electric scalar potential [2]. Immediately, in a year, the first interference experiment to verify the magnetic effect was performed [3], and this caused a burst of interest opening a long way of an extensive research both in theory and in experiment, see, e.g., [4, 5, 6, 7, 8, 9, 10, 11] and references therein; the effect conventionally bears the name of the authors of [2]. Although a great progress in understanding the Aharonov–Bohm (AB) effect has been achieved since the time of foundational works [1, 2, 3], one may agree with the authors of [12] that “the investigation and exploitation of the AB effect remain far from finished”.

It should be noted that the concern of paper [1] was in the elaboration of the consistent theory of refraction of electron rays in the framework of electron wave optics. Following thoroughly their flow of reasoning, the authors arrive at the conclusion that the medium in a spatial region outside a magnetic vortex is anisotropic and that there exist wave-optical phenomena which are due to the influence of the vector potential of the vortex. Finally, in the very end of the paper, they suggest an interference experiment to verify their findings.

An approach of paper [2] was quite different: the authors intuitively, with the use of few rather general and obvious formulas (which are even left unnumbered in the text), arrive at their suggestion of interference experiments already in the beginning of the paper. A powerful theoretical background behind this heuristic suggestion is hereafter given in Section 4. “Exact solution for scattering problems” (which exceeds in its volume two preceding sections). The background relies directly on the formalism of quantum mechanics, namely on the Schrödinger wave equation, which allows the authors to consider their effect as a scattering event. Perhaps, this was a reason, why paper [2], unlike paper [1], had received

an immediate response from the physics community.

In this respect it looks rather paradoxical that, whereas interference experiments aiming at the verification of the AB effect were numerous (see [11]), a scattering experiment aiming at the same was *never* performed. One of the motivations of the present paper was to provide an explanation for this paradox.

Let us commence by recalling the famous formula obtained by the authors of [2] for the scattering differential cross section per unit length of the magnetic vortex (see (21) and (22) in [2]):

$$\frac{d\sigma^{(\text{AB})}}{dzd\varphi} = \frac{\hbar}{2\pi p} \frac{\sin^2[e\Phi(2\hbar c)^{-1}]}{\sin^2(\varphi/2)}, \quad (1)$$

where p is the momentum of a scattered particle, e is its charge, φ is the scattering angle (with $\varphi = 0$ corresponding to the forward direction) and Φ is the flux of the magnetic vortex directed along the z -axis. A characteristic feature of (1) is a periodic dependence in the magnetic flux with the period equal to the London flux quantum, $2\pi\hbar c/|e|$. Also, one could suggest that (1) describes a purely quantum effect, since it vanishes in the formal limit of $\hbar \rightarrow 0$, or, more physically, in the limit of large enough values of the particle momentum, $p \rightarrow \infty$. However, result (1) corresponds to the idealized case of the magnetic vortex of the vanishing transverse size. Since any real magnetic vortex (either that produced by a current-carrying solenoid, or that inside a magnet) is of a nonzero transverse size, the quotient of squared sines in (1) in the real case becomes a certain function of dimensionless variable pr_c/\hbar , where r_c is a length characterizing the transverse size of the vortex. The transverse size effects were taken into account in [7], and results were found to converge with (1) in the case of $pr_c\hbar^{-1} \rightarrow 0$. As to the case of $pr_c\hbar^{-1} \gg 1$, it was only scarcely and insufficiently considered, see [6, 8], and, therefore, let us dwell on this case at more length.

It should be noted in the first place that the smallest possible values of r_c can be of order $1 \mu\text{m}$ (ferromagnetic whiskers enclosed in screening shells) [11], while, in a scattering event, the largest possible wavelengths, $\lambda = 2\pi\hbar/p$, of the lightest charged particle, electron, are of order 0.1 nm (slowly moving electrons with energies $10\text{-}100 \text{ eV}$, e.g., such as those in famous diffraction experiments by C. J. Davisson and L. H. Germer in 1927, Nobel prize in 1937). Hence, quantity $pr_c\hbar^{-1}$ takes values of order $10^4 - 10^5$ at least, and formula (1) which might be supposed to approximate the case of $pr_c\hbar^{-1} \ll 1$ has no relation to scattering of real particles. Albeit this formula looks essentially quantum, the “quantum” factor of \hbar/p appears just due to dimensional reasons, because $r_c = 0$; it is a somewhat more explicating to write \hbar/p as product $r_c \times (pr_c/\hbar)^{-1}$, where the second (dimensionless) factor is vanishingly small for all charged particles existing in nature. This was certainly understood by the authors of [2], who never proposed cross section (1) for the experimental verification.

It may seem quite reasonable to apply the quasiclassical (called also semiclassical) method of G. Wentzel, H. A. Kramers and L. Brillouin (WKB) [13, 14, 15] to solving the problem of scattering of real charged particles by an impenetrable magnetic vortex: the size of the interaction region may be estimated as being of order of r_c and larger, then condition $r_c\lambda^{-1} \gg 1$ allows one to hope that the use of the WKB method will be adequate, see, e.g., [16, 17]. However, a scepticism immediately arises: how can a purely and essentially quantum effect as is the AB one be deduced with the help of a quasiclassical (semiclassical) method which is commonly believed to give nothing else but the yield of classical theory? In this respect it should be noted that the findings of the authors of [6, 8] indicate that scattering off an impenetrable magnetic vortex in the case of $pr_c\hbar^{-1} \gg 1$ is reduced to classical scattering of point particles off an impenetrable tube with no flux.

Nevertheless, in spite of the scepticism substantiated by the results of [6, 8], we shall proceed by applying the WKB method to the solution of the scattering AB problem in the present paper. Let us recall that interference experiments aiming at the verification of the AB effect were performed with rather energetic quasiclassical electrons. We are considering electrons of the same (as in the interference experiments) and even smaller (down to 10-20 eV) energies and study direct scattering of such particles on an impenetrable magnetic vortex, using the WKB method. Whereas the authors of [6, 7, 8] employed the Dirichlet boundary condition to ensure the vortex impenetrability, we go further by employing the most general (unorthodox in the language of [6]) boundary condition that is required by the basic concepts of probability conservation and self-adjointness of the Hamilton operator. In the course of our study we shall find, in addition to the classical effect of [6, 8], a purely quantum effect that can be denoted as the scattering AB effect with real particles.

In the next section we consider a quantum-mechanical charged particle in the background of an impenetrable magnetic vortex and choose a boundary condition for the particle wave function at the vortex edge. The WKB method is used to obtain the partial waves of the solution to the Schrödinger equation in Section 3. The scattering amplitude and cross section are obtained in Section 4. The conclusions are drawn and discussed in Section 5. We relegate some details of calculation of the scattering amplitude to Appendix A, while the use of the Hamilton-Jacobi equation of motion to derive the classical effect is outlined in Appendix B.

2 Self-adjointness of the Hamilton operator and the Robin boundary condition

Defining a scalar product as $(\tilde{\chi}, \chi) = \int_{\Omega} d^3x \tilde{\chi}^* \chi$, we get, using integration by parts,

$$\begin{aligned} (\tilde{\chi}, H\chi) &\equiv \int_{\Omega} d^3x \tilde{\chi}^* \left\{ \left[-\frac{1}{2m} \left(\hbar \boldsymbol{\partial} - i \frac{e}{c} \mathbf{A} \right)^2 + V \right] \chi \right\} = \\ &= \int_{\Omega} d^3x \left\{ \left[-\frac{1}{2m} \left(\hbar \boldsymbol{\partial} - i \frac{e}{c} \mathbf{A} \right)^2 + V \right] \tilde{\chi} \right\}^* \chi - \frac{\hbar}{2m} \int_{\partial\Omega} d\boldsymbol{\sigma} \cdot \left\{ \tilde{\chi}^* \left[\left(\hbar \boldsymbol{\partial} - i \frac{e}{c} \mathbf{A} \right) \chi \right] - \right. \\ &\quad \left. - \left[\left(\hbar \boldsymbol{\partial} + i \frac{e}{c} \mathbf{A} \right) \tilde{\chi}^* \right] \chi \right\} \equiv (H^\dagger \tilde{\chi}, \chi) - i\hbar \int_{\partial\Omega} d\boldsymbol{\sigma} \cdot \mathbf{u}, \quad (2) \end{aligned}$$

where $\partial\Omega$ is a twodimensional surface bounding the threedimensional spatial region Ω ,

$$H = H^\dagger = -\frac{1}{2m} \left(\hbar \boldsymbol{\partial} - i \frac{e}{c} \mathbf{A} \right)^2 + V \quad (3)$$

is the formal expression for a general quantum-mechanical Hamilton operator in an external electromagnetic field (electric scalar potential is included in the general potential energy V), and

$$\mathbf{u} = -\frac{i}{2m} \left\{ \tilde{\chi}^* \left[\left(\hbar \boldsymbol{\partial} - i \frac{e}{c} \mathbf{A} \right) \chi \right] - \left[\left(\hbar \boldsymbol{\partial} + i \frac{e}{c} \mathbf{A} \right) \tilde{\chi}^* \right] \chi \right\}. \quad (4)$$

Operator H is Hermitian (symmetric),

$$(\tilde{\chi}, H\chi) = (H^\dagger \tilde{\chi}, \chi), \quad (5)$$

if

$$\int_{\partial\Omega} d\boldsymbol{\sigma} \cdot \mathbf{u} = 0. \quad (6)$$

The latter condition can be satisfied in various ways by imposing different boundary conditions for χ and $\tilde{\chi}$. However, among the whole variety, there may exist a possibility that a boundary condition for $\tilde{\chi}$ is the same as that for χ ; then the domain of definition of H^\dagger (set of functions $\tilde{\chi}$) coincides with that of H (set of functions χ), and operator H is called self-adjoint. Whether such a possibility exists is in general determined by the Weyl – von Neumann theory of self-adjoint operators, see, e.g., [18]. The action of a self-adjoint operator results in functions belonging to its domain of definition only, and, therefore, a multiple action

and functions of such an operator (for instance, the evolution operator) can be consistently defined.

In the present case, the problem of self-adjointness of operator H (3) is resolved by imposing a boundary condition in the form

$$\begin{aligned} & \left[\sin(\rho\pi) \mathbf{n} \cdot \left(\boldsymbol{\partial} - \frac{ie}{\hbar c} \mathbf{A} \right) \chi + \cos(\rho\pi) \frac{1}{\mathbf{n} \cdot \mathbf{x}} \chi \right] \Big|_{\mathbf{x} \in \partial\Omega} = \\ & = \left[\sin(\rho\pi) \mathbf{n} \cdot \left(\boldsymbol{\partial} - \frac{ie}{\hbar c} \mathbf{A} \right) \tilde{\chi} + \cos(\rho\pi) \frac{1}{\mathbf{n} \cdot \mathbf{x}} \tilde{\chi} \right] \Big|_{\mathbf{x} \in \partial\Omega} = 0, \end{aligned} \quad (7)$$

where \mathbf{n} is the internal unit normal to boundary $\partial\Omega$ and ρ is the real parameter (self-adjoint extension parameter). Defining current

$$\mathbf{j} = -\frac{i}{2m} \left\{ \chi^* \left[\left(\hbar \boldsymbol{\partial} - i \frac{e}{c} \mathbf{A} \right) \chi \right] - \left[\left(\hbar \boldsymbol{\partial} + i \frac{e}{c} \mathbf{A} \right) \chi^* \right] \chi \right\}, \quad (8)$$

we note that self-adjointness condition (7) results in the vanishing of the normal component of the current at the boundary:

$$\mathbf{n} \cdot \mathbf{j}|_{\mathbf{x} \in \partial\Omega} = 0. \quad (9)$$

If χ is a solution to the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \chi(t, \mathbf{x}) = H \chi(t, \mathbf{x}), \quad (10)$$

then current (8) is the probability current density satisfying the continuity equation

$$\frac{\partial}{\partial t} |\chi|^2 + \boldsymbol{\partial} \cdot \mathbf{j} = 0. \quad (11)$$

Thus self-adjointness of the Hamilton operator ensures probability conservation, $\frac{\partial}{\partial t} \int_{\Omega} d^3x |\chi|^2 = 0$.

In the following we consider a quantum-mechanical charged particle in the background of a static magnetic field, hence $V = 0$. The magnetic field is confined to an infinite tube which is classically impenetrable to the charged particle; thus, spatial region Ω is an infinite threedimensional space with the exclusion of the tube containing the magnetic flux lines, and boundary surface $\partial\Omega$ is a surface of the tube. Assuming the cylindrical symmetry of the flux tube, we use cylindrical coordinates $\mathbf{x} = (r, \varphi, z)$ with the z -axis coinciding with the axis of the tube and choose the components of the vector potential in Ω in the form

$$A_r = A_z = 0, \quad A_\varphi = \frac{\Phi}{2\pi}, \quad (12)$$

where Φ is the total magnetic flux confined in the tube. Then the Hamilton operator takes the form

$$H = -\frac{\hbar^2}{2m} \left[\frac{1}{r} \partial_r r \partial_r + \frac{1}{r^2} \left(\partial_\varphi - \frac{ie\Phi}{2\pi\hbar c} \right)^2 + \partial_z^2 \right], \quad (13)$$

and the solution to Schrödinger equation (10) with H (13) is

$$\chi(t, \mathbf{x}) = \exp(-iEt\hbar^{-1} + ip_z z \hbar^{-1}) \psi(r, \varphi), \quad (14)$$

where $\psi(r, \varphi)$ is the solution to equation

$$\left[\frac{1}{r} \partial_r r \partial_r + \frac{1}{r^2} \left(\partial_\varphi - \frac{ie\Phi}{2\pi\hbar c} \right)^2 + \frac{p^2}{\hbar^2} \right] \psi(r, \varphi) = 0, \quad (15)$$

and obeys the boundary condition, see (7),

$$[\sin(\rho\pi) r \partial_r \psi(r, \varphi) + \cos(\rho\pi) \psi(r, \varphi)]|_{r=r_c} = 0; \quad (16)$$

r_c is the tube radius, $E = (p_z^2 + p^2)(2m)^{-1}$ is the energy of the stationary scattering state, p_z and p are the momenta in the longitudinal and the radial transverse directions, respectively. One could recognize that (16) is known as the Robin boundary condition; the case of $\rho = 0$ corresponds to the Dirichlet condition (perfect reflectivity of the boundary) and the case of $\rho = 1/2$ corresponds to the Neumann condition (absolute rigidity of the boundary). It should be emphasized that parameter ρ is in general dependent on z and φ . Thus, the “number” of self-adjoint extension parameters is infinite, moreover, it is not countable but is of power of a continuum. This distinguishes the case of an extended boundary from the case of an excluded point (contact interaction) when the number of self-adjoint extension parameters is finite, being equal to n^2 for the deficiency index equal to $\{n, n\}$ (see, e.g., [19]).

The motion of the particle in the longitudinal direction is free, and our aim is to determine the particle motion in the orthogonal plane, i.e. to find a solution to (15) and (16).

3 WKB method

Let us start with the decomposition of wave function $\psi(r, \varphi)$ into partial waves

$$\psi(r, \varphi) = \sum_{n \in \mathbb{Z}} e^{in\varphi} a_n R_n(r), \quad (17)$$

where \mathbb{Z} is the set of integer numbers. The radial component of each partial wave satisfies equation

$$[\Delta_r + P_n^2(r)\hbar^{-2}]R_n(r) = 0, \quad (18)$$

where $\Delta_r = r^{-1}\partial_r r \partial_r$ and

$$P_n(r) = \sqrt{p^2 - \hbar^2(n - \mu)^2 r^{-2}}, \quad (19)$$

$\mu = e\Phi(2\pi\hbar c)^{-1}$. Note that $\psi(r, \varphi)$ (17) is periodic in the value of flux Φ with the period equal to the London flux quantum.

We present the radial component of the partial wave as $R_n(r) = \exp\left[\frac{i}{\hbar}\Sigma_n(r)\right]$, where $\Sigma_n(r)$ satisfies the Ricatti-type equation

$$\frac{i}{\hbar}\Delta_r \Sigma_n - \frac{1}{\hbar^2}(\partial_r \Sigma_n)^2 + \frac{1}{\hbar^2}P_n^2 = 0. \quad (20)$$

Expanding $\Sigma_n(r)$ into series

$$\Sigma_n(r) = \sum_{l=0}^{\infty} \left(\frac{\hbar}{ipr_c}\right)^l \Sigma_n^{(l)}(r) \quad (21)$$

and collecting terms of the same order in \hbar , we obtain the system of equations:

$$\begin{aligned} & \left[\partial_r \Sigma_n^{(0)}\right]^2 = P_n^2, \\ & 2 \left[\partial_r \Sigma_n^{(0)}\right] \left[\partial_r \Sigma_n^{(1)}\right] = -pr_c \Delta_r \Sigma_n^{(0)}, \\ & 2 \left[\partial_r \Sigma_n^{(0)}\right] \left[\partial_r \Sigma_n^{(2)}\right] = -pr_c \Delta_r \Sigma_n^{(1)} - \left[\partial_r \Sigma_n^{(1)}\right]^2, \\ & \quad \vdots \\ & 2 \left[\partial_r \Sigma_n^{(0)}\right] \left[\partial_r \Sigma_n^{(2l)}\right] = -pr_c \Delta_r \Sigma_n^{(2l-1)} - 2 \sum_{l'=1}^{l-1} \left[\partial_r \Sigma_n^{(l')}\right] \left[\partial_r \Sigma_n^{(2l-l')}\right] - \left[\partial_r \Sigma_n^{(l)}\right]^2, \\ & 2 \left[\partial_r \Sigma_n^{(0)}\right] \left[\partial_r \Sigma_n^{(2l+1)}\right] = -pr_c \Delta_r \Sigma_n^{(2l)} - 2 \sum_{l'=1}^l \left[\partial_r \Sigma_n^{(l')}\right] \left[\partial_r \Sigma_n^{(2l+1-l')}\right], \\ & \quad \vdots \end{aligned} \quad (22)$$

Consecutively solving equations beginning from the top, one can obtain $\Sigma_n(r)$ (21) up to an arbitrary order in expansion parameter $\hbar/(pr_c)$. The terms corresponding to odd l contribute to the amplitude of $R_n(r)$, while the terms corresponding to even l contribute to the phase of $R_n(r)$.

However, the power of the WKB method reveals fully itself in those cases when it suffices to take account for the first two terms, $\Sigma_n^{(0)}(r)$ and $\Sigma_n^{(1)}(r)$, only. It should be noted that equation (18), or (20), has its analogue in optics, describing the propagation of waves in media [20]: quotient $P_n(r)/p$ corresponds to the index

of refraction. Terms $\Sigma_n^{(l)}(r)$ ($l \geq 2$) are negligible, if the variation of the refraction index along the ray of propagation becomes appreciable on the distances which are much larger than the effective wavelength,

$$|P_n^{-1} \boldsymbol{\partial} \cdot (\mathbf{n} P_n)|^{-1} \gg \hbar |P_n|^{-1}, \quad (23)$$

where \mathbf{n} points in the direction of the ray which in our case is orthogonal to the boundary. In view of (19) this yields condition

$$\frac{\hbar |p^2 r^2|}{|p^2 r^2 - \hbar^2 (n - \mu)^2|^{3/2}} \ll 1, \quad (24)$$

which is satisfied in the quasiclassical region,

$$pr \gg \hbar |n - \mu|, \quad pr \gg \hbar, \quad (25)$$

as well as in the deeply nonclassical region

$$pr \ll \hbar |n - \mu|, \quad |n - \mu| \gg 1. \quad (26)$$

Solving the first two equations in (22), we get

$$\Sigma_n^{(0)}(r) = \pm \int^r dr P_n(r), \quad \Sigma_n^{(1)}(r) = -\frac{1}{2} pr_c \ln[r P_n(r)]. \quad (27)$$

It should be noted that the second equation in (22) is nothing more but the continuity equation, $\boldsymbol{\partial} \cdot \mathbf{j}_n$, for the partial wave current,

$$\mathbf{j}_n = -\frac{i\hbar}{2m} [e^{-in\varphi} R_n^* (\boldsymbol{\partial} e^{in\varphi} R_n) - (\boldsymbol{\partial} e^{-in\varphi} R_n^*) e^{in\varphi} R_n],$$

in the quasiclassical region.

Actually, the WKB approximation (i.e. the neglect of $\Sigma_n^{(l)}(r)$ with $l \geq 2$) is efficient in a much more extended regions than those given by (25) and (26). The WKB approximation is not valid in the vicinity of $r = r_t$ which is defined by

$$P_n(r_t) = 0, \quad (28)$$

this point is the turning point of a classical trajectory of the particle. In view of (27), the two linearly independent solutions to (18) in the quasiclassical region are

$$R_{n,\text{out}}^{(\pm)}(r) = (r P_n)^{-1/2} \exp \left(\pm \frac{i}{\hbar} \int_{r_t}^r dr P_n \right), \quad r > r_t, \quad (29)$$

while the two linearly independent solutions to (18) in the deeply nonclassical region are

$$R_{n,\text{in}}^{(\pm)}(r) = (r\Pi_n)^{-1/2} \exp\left(\pm \frac{1}{\hbar} \int_{r_t}^r dr \Pi_n\right), \quad r < r_t, \quad (30)$$

where

$$\Pi_n(r) = \sqrt{\hbar^2(n - \mu)^2 r^{-2} - p^2}. \quad (31)$$

The crucial question is how oscillating solutions (29) in the outer region go over to solutions (30) in the inner region and vice versa. To answer this question, one has to find a solution in the vicinity of the turning point, where $P_n(r) = p\sqrt{\frac{2}{r_t}(r - r_t)}$ and $\Pi_n(r) = p\sqrt{\frac{2}{r_t}(r_t - r)}$, and to match it with the solutions in the inner and the outer regions. But a more instructive way, as stated in [17], is to consider formally $R_n(r)$ as a function of complex variable r and to perform transitions along paths in the complex plane, where condition (24) is satisfied. Namely, a path goes along the real positive semiaxis up to the left vicinity of the turning point, circumvents it along a semicircle in the upper or lower half-plane, and then goes again along the real positive semiaxis. For the transition from the inner to the outer region we have

$$\Pi_n(r) \rightarrow P_n(r)e^{\pm i\pi/2}, \quad (32)$$

where the (\pm) sign corresponds to the path in the upper or lower half-plane, and

$$R_{n,\text{in}}^{(+)}(r) \rightarrow R_{n,\text{out}}^{(+)}(r)e^{-i\pi/4} + R_{n,\text{out}}^{(-)}(r)e^{i\pi/4}. \quad (33)$$

$R_{n,\text{in}}^{(-)}(r)$, in contrast to $R_{n,\text{in}}^{(+)}(r)$, is divergent in the inner region. For the transition from the outer to the inner region we have

$$P_n(r) \rightarrow \Pi_n(r)e^{\pm i\pi/2}, \quad (34)$$

and

$$\frac{i}{2} \left[R_{n,\text{out}}^{(+)}(r)e^{-i\pi/4} - R_{n,\text{out}}^{(-)}(r)e^{i\pi/4} \right] \rightarrow R_{n,\text{in}}^{(-)}(r), \quad (35)$$

where the account is taken for the fact that $R_{n,\text{in}}^{(-)}(r)$ is real.

Boundary condition (16) in terms of partial waves takes form

$$[\sin(\rho\pi)r\partial_r R_n(r) + \cos(\rho\pi)R_n(r)]|_{r=r_c} = 0. \quad (36)$$

In the case $r_t < r_c \leq r$ we choose

$$R_n(r) = \left[R_{n,\text{out}}^{(-)}(r) - C_n(r_c, \rho) R_{n,\text{out}}^{(+)}(r) \right] e^{i\pi/4}, \quad (37)$$

where the factor of $\exp(i\pi/4)$ is inserted for future convenience. Coefficient $C_n(r_c, \rho)$ is determined from condition (36):

$$C_n(r_c, \rho) = \frac{R_{n,\text{out}}^{(-)}(r_c) \cot(\rho\pi) + [r\partial_r \ln R_{n,\text{out}}^{(-)}(r)]|_{r=r_c}}{R_{n,\text{out}}^{(+)}(r_c) \cot(\rho\pi) + [r\partial_r \ln R_{n,\text{out}}^{(+)}(r)]|_{r=r_c}} =$$

$$= \exp \left\{ -\frac{2i}{\hbar} \int_{r_t}^{r_c} dr P_n - 2i \arctan \left[\frac{r_c P_n(r_c) \hbar^{-1}}{\cot(\rho\pi) - \frac{1}{2} \frac{p^2}{P_n^2(r_c)}} \right] \right\}, \quad r_c > r_t; \quad (38)$$

note that the explicit form of the turning point (i.e. the solution to (28)) is evidently $r_t = \hbar|n - \mu|/p$. In the case $r \leq r_c < r_t$ we choose

$$R_n(r) = R_{n,\text{in}}^{(+)}(r) - C_n(r_c, \rho) \left[R_{n,\text{in}}^{(-)}(r) + c R_{n,\text{in}}^{(+)}(r) \right], \quad (39)$$

where coefficient $C_n(r_c, \rho)$ is determined from condition (36):

$$C_n(r_c, \rho) = \frac{R_{n,\text{in}}^{(+)}(r_c)}{R_{n,\text{in}}^{(-)}(r_c) + c R_{n,\text{in}}^{(+)}(r_c)} \times$$

$$\times \frac{\cot(\rho\pi) + [r\partial_r \ln R_{n,\text{in}}^{(+)}(r)]|_{r=r_c}}{\cot(\rho\pi) + \left\{ r\partial_r \ln [R_{n,\text{in}}^{(-)}(r) + c R_{n,\text{in}}^{(+)}(r)] \right\}|_{r=r_c}}, \quad r_c < r_t,$$

and constant c is to be fixed before long. Using correspondence rules (33) and (35), we continue $R_n(r)$ (39) to the case $r_c < r_t < r$ as

$$R_n(r) = R_{n,\text{out}}^{(+)}(r) e^{-i\pi/4} + R_{n,\text{out}}^{(-)}(r) e^{i\pi/4} -$$

$$- C_n(r_c, \rho) \left\{ \frac{i}{2} \left[R_{n,\text{out}}^{(+)}(r) e^{-i\pi/4} - R_{n,\text{out}}^{(-)}(r) e^{i\pi/4} \right] + c \left[R_{n,\text{out}}^{(+)}(r) e^{-i\pi/4} + R_{n,\text{out}}^{(-)}(r) e^{i\pi/4} \right] \right\}.$$

The converging wave, $R_{n,\text{out}}^{(-)}(r)$, should disappear from the part with coefficient $C_n(r_c, \rho)$, cf. (37). This fixes the constant, $c = i/2$, yielding in the case $r_c < r_t < r$

$$R_n(r) = R_{n,\text{out}}^{(+)}(r) e^{-i\pi/4} + R_{n,\text{out}}^{(-)}(r) e^{i\pi/4} - C_n(r_c, \rho) R_{n,\text{out}}^{(+)}(r) e^{i\pi/4}, \quad (40)$$

with

$$C_n(r_c, \rho) = \exp \left(\frac{2}{\hbar} \int_{r_t}^r dr \Pi_n \right) \times$$

$$\times \left[\frac{\cot(\rho\pi) + \frac{1}{2} \frac{p^2}{\Pi_n^2(r_c)} - r_c \Pi_n(r_c) \hbar^{-1}}{\cot(\rho\pi) + \frac{1}{2} \frac{p^2}{\Pi_n^2(r_c)} + r_c \Pi_n(r_c) \hbar^{-1}} + \frac{i}{2} \exp \left(\frac{2}{\hbar} \int_{r_t}^{r_c} dr \Pi_n \right) \right]^{-1}, \quad r_c < r_t. \quad (41)$$

We can rewrite (37) in the case $r_t < r_c \leq r$ as

$$R_n(r) = R_{n,\text{out}}^{(+)}(r)e^{-i\pi/4} + R_{n,\text{out}}^{(-)}(r)e^{i\pi/4} - [e^{-i\pi/2} + C_n(r_c, \rho)] R_{n,\text{out}}^{(+)}(r)e^{i\pi/4}. \quad (42)$$

We conclude this section by stating that (40) and (42) give the radial components of partial waves of $\psi(r, \varphi)$ (17) in the quasiclassical region in the WKB approximation.

4 Scattering amplitude and cross section

Going over to asymptotics $pr\hbar^{-1} \rightarrow \infty$, where

$$rP_n(r) = pr + O[\hbar^2(pr)^{-1}], \quad \int_{r_t}^r dr P_n = pr - \frac{\hbar}{2}|n - \mu|\pi + O[\hbar^2(pr)^{-1}], \quad (43)$$

we obtain

$$\psi(r, \varphi) = \psi^{(0)}(r, \varphi) + \psi^{(c)}(r, \varphi), \quad (44)$$

where

$$\psi^{(0)}(r, \varphi) = \frac{2}{\sqrt{pr}} \sum_{n \in \mathbb{Z}} e^{in\varphi} a_n \cos \left(pr\hbar^{-1} - \frac{1}{2}|n - \mu|\pi - \frac{1}{4}\pi \right), \quad (45)$$

and

$$\begin{aligned} \psi^{(c)}(r, \varphi) = -\frac{e^{i(pr\hbar^{-1} + \pi/4)}}{\sqrt{pr}} \left\{ \sum_{|n - \mu| \leq pr_c/\hbar} e^{in\varphi} a_n e^{-\frac{i}{2}|n - \mu|\pi} [e^{-i\pi/2} + C_n(r_c, \rho)] + \right. \\ \left. + \sum_{|n - \mu| > pr_c/\hbar} e^{in\varphi} a_n e^{-\frac{i}{2}|n - \mu|\pi} C_n(r_c, \rho) \right\}. \quad (46) \end{aligned}$$

To determine coefficient a_n , let us recall the asymptotics of the partial wave decomposition of a plane wave,

$$\begin{aligned} e^{ipr\hbar^{-1} \cos \varphi} &= \frac{1}{\sqrt{2\pi pr\hbar^{-1}}} \sum_{n \in \mathbb{Z}} e^{in\varphi} [e^{i(pr\hbar^{-1} - \pi/4)} + e^{in|\pi} e^{-i(pr\hbar^{-1} - \pi/4)}] = \\ &= \sqrt{\frac{2\pi\hbar}{pr}} \left[\Delta(\varphi) e^{i(pr\hbar^{-1} - \pi/4)} + \Delta(\varphi - \pi) e^{-i(pr\hbar^{-1} - \pi/4)} \right], \quad (47) \end{aligned}$$

where $\Delta(\varphi) = (2\pi)^{-1} \sum_{n \in \mathbb{Z}} e^{in\varphi}$ is the angular delta-function, $\Delta(\varphi + 2\pi) = \Delta(\varphi)$.

The asymptotics of the plane wave is naturally interpreted as a superposition

of two cylindrical waves: the diverging one, $e^{ipr\hbar^{-1}}/\sqrt{r}$ going in the forward, $\varphi = 0$, direction and the converging one, $e^{-ipr\hbar^{-1}}/\sqrt{r}$, coming from the backward, $\varphi = \pi$, direction. Namely the converging cylindrical wave should be present without distortions in the asymptotics of wave function $\psi(r, \varphi)$ (44), whereas the diverging cylindrical wave is distorted and differs from that in (47); actually, this is the condition that $\psi(r, \varphi)$ be the scattering state solution. Equating terms before $e^{-ipr\hbar^{-1}}/\sqrt{r}$ in $\psi(r, \varphi)$ to the terms before $e^{-ipr\hbar^{-1}}/\sqrt{r}$ in the first line of (47), we get

$$a_n = \frac{1}{\sqrt{2\pi\hbar^{-1}}} \exp \left[i \left(|n| - \frac{1}{2}|n - \mu| \right) \pi \right]. \quad (48)$$

As a result, we obtain the following expression for the r_c -independent part of the wave function

$$\psi^{(0)}(r, \varphi) = \psi_0^{(0)}(r, \varphi) + f_0(p, \varphi) \frac{e^{i(pr\hbar^{-1} + \pi/4)}}{\sqrt{r}}, \quad (49)$$

where

$$\begin{aligned} \psi_0^{(0)}(r, \varphi) &= e^{ipr\hbar^{-1} \cos \varphi} e^{i\mu[\varphi - \text{sgn}(\varphi)\pi]} = \\ &= \sqrt{\frac{2\pi\hbar}{pr}} \left[e^{-i\mu \text{sgn}(\varphi)\pi} \Delta(\varphi) e^{i(pr\hbar^{-1} - \pi/4)} + \Delta(\varphi - \pi) e^{-i(pr\hbar^{-1} - \pi/4)} \right], \end{aligned} \quad (50)$$

is the incident wave and

$$f_0(p, \varphi) = \frac{\sin(\mu\pi)}{\sqrt{2\pi p\hbar^{-1}}} \sum_{n \in \mathbb{Z}} \text{sgn}(n - \mu) e^{in\varphi} = i \frac{\sin(\mu\pi)}{\sqrt{2\pi p\hbar^{-1}}} \frac{e^{i(\llbracket \mu \rrbracket + \frac{1}{2})\varphi}}{\sin(\varphi/2)} \quad (51)$$

is the scattering amplitude [2]; the sign function is $\text{sgn}(u) = \pm 1$ at $u \gtrless 0$, $\llbracket u \rrbracket$ denotes the integer part of quantity u (i.e. the integer which is less than or equal to u) and it is implied that $-\pi < \varphi < \pi$. The squared absolute value of (51) yields differential cross section (1). Incident wave (50) differs from the plane wave: the distortions are due to the long-range nature of the vector potential outside the flux tube. The apparent divergence of amplitude (51) in the forward direction is a phantom, because (51) is valid under condition $\sqrt{pr/\hbar} |\sin(\varphi/2)| \gg 1$, whereas, otherwise, the divergence is absent and the discontinuity in incident wave (50) at $\varphi \rightarrow \pm 0$ is cancelled, yielding $\psi^{(0)}(r, \varphi)$ which is continuous and differentiable at $\varphi = 0$, see, e.g., [21, 22] and references therein. Amplitude f_0 (51) is of order $\sqrt{r_c} O \left[\sqrt{\hbar/(pr_c)} \right]$ and is negligible as compared to the contribution from the r_c -dependent part of the wave function, which can be presented in the following form:

$$\psi^{(c)}(r, \varphi) = [f_1(p, \varphi) + f_2(p, \varphi) + f_3(p, \varphi)] \frac{e^{i(pr\hbar^{-1} + \pi/4)}}{\sqrt{r}}, \quad (52)$$

where

$$f_1(p, \varphi) = \frac{i}{\sqrt{2\pi p \hbar^{-1}}} \sum_{|n-\mu| \leq pr_c/\hbar} e^{in\varphi} e^{i(|n|-|n-\mu|)\pi}, \quad (53)$$

$$f_2(p, \varphi) = -\frac{1}{\sqrt{2\pi p \hbar^{-1}}} \sum_{|n-\mu| \leq pr_c/\hbar} e^{in\varphi} e^{i(|n|-|n-\mu|)\pi} C_n(r_c, \rho), \quad (54)$$

$$f_3(p, \varphi) = -\frac{1}{\sqrt{2\pi p \hbar^{-1}}} \sum_{|n-\mu| > pr_c/\hbar} e^{in\varphi} e^{i(|n|-|n-\mu|)\pi} C_n(r_c, \rho). \quad (55)$$

We show in Appendix A (see (A.22)) that amplitude f_3 (55), although exceeding amplitude f_0 (51), is still negligible, being of order $\sqrt{r_c} O[\hbar^{1/6} (pr_c)^{-1/6}]$. Amplitude f_2 (54) is of nonnegligible value which is calculated in Appendix A:

$$f_2(p, \varphi) = -\sqrt{\frac{r_c}{2}} |\sin(\varphi/2)| \exp \left\{ -2ipr_c \hbar^{-1} |\sin(\varphi/2)| + i\mu[\varphi - \text{sgn}(\varphi)\pi] - i\pi/4 \right\} \times \\ \times \exp \left\{ -2i \arctan \left[\frac{2pr_c \hbar^{-1} |\sin^3(\varphi/2)|}{2 \cot(\rho\pi) \sin^2(\varphi/2) - 1} \right] \right\}. \quad (56)$$

Amplitude f_1 (53) is a finite sum of geometric progression, which is straightforwardly calculated:

$$f_1(p, \varphi) = i \sqrt{\frac{2\hbar}{\pi p}} e^{i(\llbracket \mu \rrbracket + \frac{1}{2})\varphi} \frac{\sin(s_c \varphi/2)}{\sin(\varphi/2)} \cos(\mu\pi + s_c \varphi/2) \quad (57)$$

in the case

$$\llbracket pr_c \hbar^{-1} + \mu \rrbracket - \llbracket \mu \rrbracket = \llbracket pr_c \hbar^{-1} - \mu \rrbracket + \llbracket \mu \rrbracket + 1 = s_c, \quad (58)$$

or

$$f_1(p, \varphi) = i \sqrt{\frac{2\hbar}{\pi p}} e^{i(\llbracket \mu \rrbracket + \frac{1}{2} \mp \frac{1}{2})\varphi} \left\{ \frac{\sin[(s_c + 1/2)\varphi/2]}{\sin(\varphi/2)} \cos[\mu\pi + (s_c + 1/2)\varphi/2] - \right. \\ \left. - \tan(\varphi/4) \pm i \right\} \quad (59)$$

in the case

$$\llbracket pr_c \hbar^{-1} + \mu \rrbracket - \llbracket \mu \rrbracket - \frac{1}{2} \pm \frac{1}{2} = \llbracket pr_c \hbar^{-1} - \mu \rrbracket + \llbracket \mu \rrbracket + \frac{1}{2} \mp \frac{1}{2} = s_c. \quad (60)$$

Since $s_c \approx pr_c \hbar^{-1} \gg 1$, amplitude f_1 is strongly peaked in the forward direction (actually as a smoothed delta-function), whereas amplitude f_2 is vanishing in the forward direction. Therefore, the interference between the amplitudes, $f_1^* f_2 +$

$f_2^* f_1$, is vanishing in the same manner as $\sqrt{|\varphi|} \Delta(\varphi) = 0$. The differential cross section of the scattering process is hence a sum of two terms:

$$\frac{d\sigma_2}{dzd\varphi} \equiv |f_2(p, \varphi)|^2 = \frac{r_c}{2} \left| \sin \frac{\varphi}{2} \right| \quad (61)$$

and

$$\frac{d\sigma_1}{dzd\varphi} \equiv |f_1(p, \varphi)|^2 = 4r_c \Delta_{\frac{pr_c}{2\hbar}}(\varphi) \cos^2 \left[\left(pr_c \varphi + \frac{e}{c} \Phi \right) (2\hbar)^{-1} \right], \quad (62)$$

where we have recalled that $\mu = e\Phi(2\pi\hbar c)^{-1}$ and introduced function

$$\Delta_y(\varphi) = \frac{1}{4\pi y} \frac{\sin^2(y\varphi)}{\sin^2(\varphi/2)} \quad (-\pi < \varphi < \pi), \quad (63)$$

which at $y \gg 1$ can be regarded as a regularized (smoothed) delta-function,

$$\lim_{y \rightarrow \infty} \Delta_y(\varphi) = \Delta(\varphi), \quad \Delta_y(0) = \frac{y}{\pi}, \quad \int_{-\pi}^{\pi} d\varphi \Delta_y(\varphi) = 1 + O(y^{-2}).$$

Term (61) which is independent of the self-adjoint extension parameter, the magnetic flux and even the particle momentum is obtainable in the framework of classical theory, for instance, with the use of the Hamilton-Jacobi equation of motion, see Appendix B. Amplitude f_2 (54) can be rewritten as

$$f_2(p, \varphi) = -\frac{i}{\sqrt{2\pi p \hbar^{-1}}} \sum_{|n-\mu| \leq pr_c/\hbar} \exp \left\{ i \left[n\varphi + 2\delta_n^{\text{WKB}}(pr_c \hbar^{-1}) - 2\omega_n(pr_c \hbar^{-1}, \rho) \right] \right\}, \quad (64)$$

where

$$\begin{aligned} \delta_n^{\text{WKB}}(pr_c \hbar^{-1}) &= \frac{1}{2} n \text{sgn}(n - \mu) \pi - \frac{1}{\hbar} \int_{\infty}^{r_c} dr (P_n - p) - \frac{1}{\hbar} pr_c - \frac{1}{4} \pi = \\ &= \frac{1}{2} \mu \text{sgn}(n - \mu) \pi - \xi_n(pr_c \hbar^{-1}) - \frac{1}{4} \pi \end{aligned} \quad (65)$$

is the WKB phase shift, $\xi_n(s)$ and $\omega_n(s, \rho)$ are given by (A.2) and (A.3) in Appendix A. The derivative of the WKB phase shift, $\frac{\partial}{\partial n} \delta_n^{\text{WKB}}$, can be related to the classical reflection angle (equal to the classical incidence angle), $\theta_{r_c} - \theta_{\infty}$, see (B.10) in Appendix B:

$$\left(\frac{\partial}{\partial n} \delta_n^{\text{WKB}} \right) \Big|_{n=\mu+\alpha/\hbar} = \theta_{r_c} - \theta_{\infty} + \frac{1}{2} \text{sgn}(\alpha) \pi. \quad (66)$$

Thus, the condition of stationarity of the phase in (64),

$$\varphi = -2 \left(\frac{\partial}{\partial n} \delta_n^{\text{WKB}} \right) \Big|_{n=n_0} \quad (67)$$

($\frac{\partial}{\partial n} \omega_n$ is negligible, see Appendix A), coincides in its form with the classical relation between scattering angle φ and reflection angle $\theta_{r_c} - \theta_\infty$, see (B.12) in Appendix B, and the classical impact parameter can be introduced as

$$b = -\hbar(n_0 - \mu)p^{-1}; \quad (68)$$

note also that the first equation in (22) is equivalent to the equation determining the squared r -derivative of the Hamilton-Jacobi action, see (B.5) in Appendix B.

However, the merits of the WKB method are not restricted to determining the WKB phase shift, i.e. to the only description of classical elastic reflection. The use of the method has also allowed us to find term (62) which is generically independent of the self-adjoint extension parameter but depends on both the magnetic flux and the particle momentum. This term describes the purely quantum effect of diffraction. It should be noted that the quantum effect is well separated in the scattering angle from the classical one: namely, diffraction is in the forward direction where classical reflection is absent. Although the range of angles where diffraction is to be observed is quite narrow, the whole (i.e. integrated over $-\pi < \varphi < \pi$) contribution of the diffraction effect to the total cross section is the same as that of the reflection effect:

$$\frac{d\sigma_1}{dz} = \frac{d\sigma_2}{dz} = 2r_c, \quad (69)$$

and the total cross section is twice the classical one; the latter is needed for the optical theorem to be maintained (see [23, 24]).

It should be noted that the AB scattering amplitude (see (51)) and, consequently, cross section (1) were obtained by imposing the condition of regular behavior for the particle wave function at the location of the singular ($r_c = 0$) magnetic vortex [2]. However, the regularity condition is not the most general one that is required by probability conservation. The most general condition in the case when a spatial dimension along the vortex is ignored involves four self-adjoint extension parameters (the deficiency index is $\{2,2\}$) [25, 26]. The dependence on the self-adjoint extension parameters enters into terms which are added to amplitude f_0 (51). Therefore, the cross section depends on four arbitrary parameters or, if a longitudinal dimension is taken into account, on four arbitrary functions of z . If an invariance under rotations around the vortex is assumed to be a physical requirement, then the cross section still remains to be dependent on two arbitrary functions. The problem of seeking some, if any, sense

for this arbitrariness is of purely academic interest, since the case of a vortex with $r_c = 0$ has no relation to physics reality.

Returning to the physical case of a vortex with $r_c > 0$, we would like to emphasize that the classical initial equation (see (B.2) in Appendix B), as well as the quantum-mechanical one (see (10) with (13)), involves the vector potential (see (12)). However, in classical theory, this potential has no physical consequences, as is demonstrated, for example, in Appendix B. One may agree with the authors of [12] that “the AB effect was already implicit in the 1926 Schrödinger equation”, but it has taken more than eight decades before the physical consequence of the vector potential from the Schrödinger equation is at last pointed out, see (62).

5 Discussion and conclusion

The AB effect, as well as quantization of the flux trapped in superconductors, the Van Hove singularities in the excitation spectra of crystals etc, explicates an importance of topological concepts for quantum physics [27]. The AB phase, $e\Phi/(\hbar c)$, which is acquired by the charged-particle wave function after the particle has encircled a magnetic vortex is certainly invariant as the particle momentum varies arbitrarily. Nonetheless, a question always is how to observe this phase in real experiments, and what are physical restrictions on the particle momentum in these experiments. For instance, interference experiments to verify the AB effect involve electrons of energies restricted to the range of 10-100 KeV [8]. Interference is immanently a wave-optical phenomenon which is described in quantum theory indirectly, through reference to wave optics (a quantum-mechanical particle possesses de Broglie’s wavelength and thus is subject to the laws of wave optics). Meanwhile, quantum theory provides a direct description of the AB effect as a scattering event. A scattering experiment with particles of the wavelength exceeding the vortex radius is hardly realistic, and that is why we are considering a scattering event with particles of the wavelength which is less than the vortex radius.

By using the WKB method in the present paper, we obtain the scattering differential cross section, $d\sigma/(dzd\varphi)$, consisting of two parts: one (61) describes the classical effect of elastic reflection from the vortex and another one (62) describes the quantum effect of diffraction on the vortex. The effects are equal each other, when integrated over the whole range of the scattering angle, see (69). It should be noted that, although the amplitude yielding the classical effect depends on the parameter of the Robin boundary condition (ρ), the flux of the vortex (Φ) and the particle momentum (p), see (56), all the dependence disappears in the differential cross section, see (61); certainly, this cross section can be obtained with the use of purely classical methods (the Hamilton-Jacobi

equation of motion).

The amplitude yielding the quantum effect is generically independent of the parameter of the Robin boundary condition, see (57)-(60). This signifies that diffraction (to be more precise, the Fraunhofer diffraction, i.e. the diffraction in almost parallel rays) is invariant as long as probability is conserved. Owing to diffraction the AB effect persists in scattering of real particles, e.g., electrons of energies in the range from 10 eV to 100 KeV. The classical effect opens a window in the forward direction, where the AB effect is to be observed as a fringe shift in the diffraction pattern exhibited by the scattering differential cross section, see (62).

We present the window for the observation of the AB effect on Fig. 1, where the scattering differential cross section and the scattering angle are normalized in such a way that the plot remains actually the same for $50 < pr_c(2\pi\hbar)^{-1} < \infty$; recall that the present-day realistic restriction is $10^4 < pr_c(2\pi\hbar)^{-1} < \infty$. The classical effect given by (61) is unobservable at $|\varphi| < 10\pi\hbar(pr_c)^{-1}$, whereas the diffraction oscillations given by (62) are unobservable on the classical background at $2\arcsin[10\hbar/(\pi pr_c)]^{1/3} < |\varphi| < \pi$; at $10\pi\hbar(pr_c)^{-1} < |\varphi| < 2\arcsin[10\hbar/(\pi pr_c)]^{1/3}$, both effects are unobservable being indistinguishable from the $|f_3|^2$ -background which is of order of $[\hbar/(pr_c)]^{1/3}$, see (A.22) in Appendix A. A gate for particles propagating in the strictly forward ($\varphi = 0$) direction is opened when the vortex flux equals an integer times the London flux: more than 90% of the diffraction cross section is accumulated in a peak centred at $\varphi = 0$ and having width $2\pi\hbar/(pr_c)$, see the dashed line on Fig. 1. The gate for the strictly forward propagation of particles is closed when the vortex flux equals a half-integer times the London flux: a dip at $\varphi = 0$ is surrounded by two symmetric peaks accumulating more than 85% of the diffraction cross section (each peak having width $2\pi\hbar/(pr_c)$), see the solid line on Fig. 1. Thus, the classically forbidden propagation of particles in the strictly forward direction is controlled by altering the amount of the vortex flux.

This scattering AB effect was never observed experimentally, since a detailed proposal for its observation was just recently elaborated [28]. Such an experiment is in our opinion of fundamental significance and seems to be quite feasible with the present-day facilities providing for sources of bright and yet coherent electron beams.

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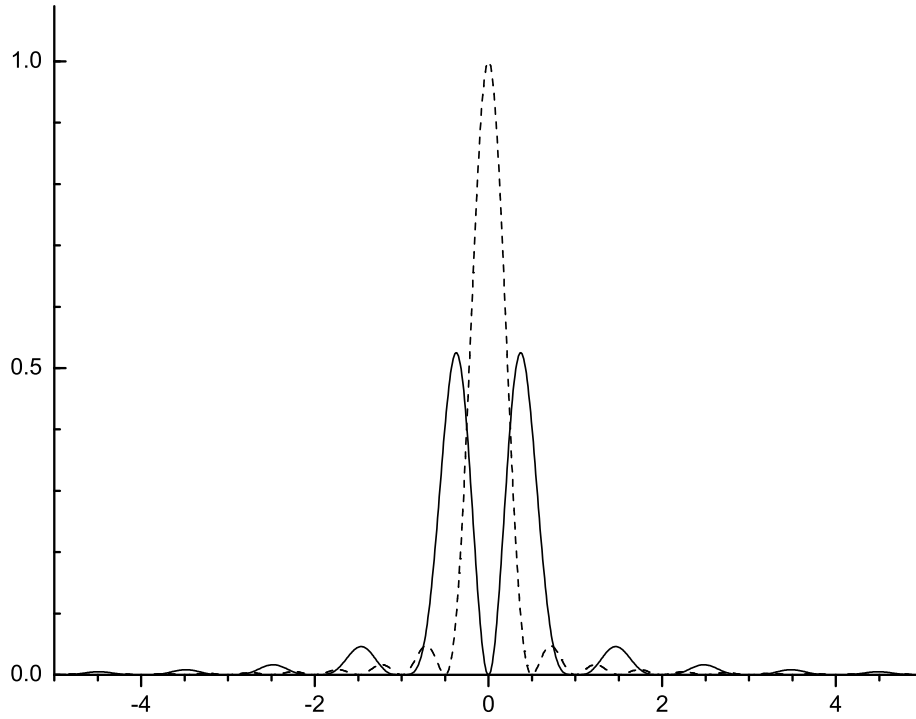


Figure 1: $\frac{d\sigma}{dzd\varphi} / \left(\frac{pr_c}{h} \frac{d\sigma}{dz} \right)$ is along the ordinate axis and $\varphi pr_c/h$ is along the abscissa axis ($\hbar = 2\pi\hbar$). Dashed and solid lines correspond to cases $\Phi = nhc/e$ and $\Phi = (n + 1/2)hc/e$, respectively ($n \in \mathbb{Z}$). The area under one central peak by the dashed line ($|\varphi| < h/(2pr_c)$) is 0.4514119 ± 0.0000002 , and the area under two central peaks by the solid line ($|\varphi| < h/(pr_c)$) is 0.4278549 ± 0.0000013 .

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Appendix A. Calculation of amplitudes f_2 and f_3

We present amplitude f_2 (54) in the form

$$f_2(p, \varphi) = -\frac{1}{\sqrt{2\pi p \hbar^{-1}}} \times \sum_{|n-\mu| \leq pr_c/\hbar} \exp \{i [n\varphi + \mu \operatorname{sgn}(n - \mu)\pi - 2\xi_n(pr_c \hbar^{-1}) - 2\omega_n(pr_c \hbar^{-1}, \rho)]\}, \quad (\text{A.1})$$

where, see (38),

$$\xi_n(s) = \frac{1}{\hbar} \int_{\hbar|n-\mu|/p}^{r_c} dr P_n = \sqrt{s^2 - (n - \mu)^2} - |n - \mu| \arccos(|n - \mu|/s), \quad (\text{A.2})$$

$$w_n(s, \rho) = \arctan \left[\frac{\sqrt{s^2 - (n - \mu)^2}}{\cot(\rho\pi) - \frac{1}{2} \frac{s^2}{s^2 - (n - \mu)^2}} \right], \quad (\text{A.3})$$

and we have introduced dimensionless variable $s = pr_c/\hbar$. To calculate the sum in (A.1) in asymptotics $s \gg 1$, we use the Poisson summation formula

$$\sum_{|n-\mu| \leq s} e^{i\eta(n,s)} = \sum_{l \in \mathbb{Z}} \int_{-s_-}^{s_+} dn \exp \{i[\eta(n, s) - 2\pi nl]\} + \frac{1}{2} e^{i\eta(s_+, s)} + \frac{1}{2} e^{i\eta(-s_-, s)}, \quad (\text{A.4})$$

where $s_{\pm} = \llbracket s \pm \mu \rrbracket$. If $s_+ + s_- \gg 1$ and $\eta(n, s)$ is convex upwards, $\frac{\partial^2 \eta(n, s)}{\partial n^2} < 0$, on interval $-s_- < n < s_+$, then only a finite number of terms in the series on the right-hand side of (A.4) contributes to the leading asymptotics at $s \gg 1$, and one can use the method of stationary phase for its evaluation. Namely, if equation

$$\left[\frac{\partial}{\partial n} \eta(n, s) \right] \Big|_{n=n_j} - 2\pi l_j = 0 \quad (\text{A.5})$$

determines a stationary point inside the interval, $-s_- < n_j < s_+$, for some values of l denoted by l_j , then

$$\sum_{|n-\mu| \leq s} e^{i\eta(n,s)} = \sum_{l_j} \exp \{i[\eta(n_j, s) - 2\pi n_j l_j]\} \left\{ \frac{2\pi e^{-i\pi/2}}{-\left[\frac{\partial^2}{\partial n^2} \eta(n, s) \right] \Big|_{n=n_j}} \right\}^{1/2} + O(1). \quad (\text{A.6})$$

In the present case we have

$$\eta(n, s) = n\varphi + \mu \operatorname{sgn}(n - \mu)\pi - 2[\xi_n(s) + \omega_n(s, \rho)]. \quad (\text{A.7})$$

Differentiating (A.2) and (A.3) over n , we obtain

$$\frac{\partial}{\partial n} \xi_n(s) = -\operatorname{sgn}(n - \mu) \arccos(|n - \mu|/s) \quad (\text{A.8})$$

and

$$\frac{\partial}{\partial n} \omega_n(s, \rho) = -\frac{\frac{n-\mu}{s} \left[1 - \left(\frac{n-\mu}{s}\right)^2\right]^{1/2} \left\{ \left[1 - \left(\frac{n-\mu}{s}\right)^2\right] \cot(\rho\pi) - \frac{3}{2} \right\}}{\left\{ \left[1 - \left(\frac{n-\mu}{s}\right)^2\right] \cot(\rho\pi) - \frac{1}{2} \right\}^2 + s^2 \left[1 - \left(\frac{n-\mu}{s}\right)^2\right]^3}. \quad (\text{A.9})$$

(A.9) is negligible as compared to (A.8), being of order $O(|n-\mu|/s^3)$ at $|n-\mu| \ll s$, $s \gg 1$, where the WKB approximation is valid, see (25). Therefore equation (A.5) takes form

$$\varphi + 2\operatorname{sgn}(n_j - \mu) \arccos(|n_j - \mu|/s) - 2\pi l_j = 0. \quad (\text{A.10})$$

Choosing the range for φ as $-\pi < \varphi < \pi$, we find that a solution to (A.10) exists at $l_j = 0$ only. Denoting the solution by $n_j = n_0$, we obtain

$$n_0 - \mu = -s \operatorname{sgn}(\varphi) \cos(\varphi/2), \quad (\text{A.11})$$

and, hence,

$$\left[\frac{\partial^2}{\partial n^2} \eta(n, s) \right] \Big|_{n=n_0} = -\frac{2}{s |\sin(\varphi/2)|}. \quad (\text{A.12})$$

We see that the term corresponding to $l = 0$ in the right-hand side of (A.4) is of order $O(\sqrt{s})$, whereas all other terms are of order $O(1)$. As a result, we obtain expression (56).

Amplitude f_3 (55) is presented in the form

$$\begin{aligned} f_3(p, \varphi) = & -\frac{1}{\sqrt{2\pi p \hbar^{-1}}} \sum_{|n-\mu|>s} e^{i[n\varphi + \mu \operatorname{sgn}(n-\mu)\pi]} \times \\ & \times \exp \left[2\sqrt{(n-\mu)^2 - s^2} - 2|n-\mu| \operatorname{arccosh}(|n-\mu|/s) \right] \times \\ & \times \left\{ \frac{\cot(\rho\pi) + \frac{1}{2} \frac{s^2}{(n-\mu)^2 - s^2} - \sqrt{(n-\mu)^2 - s^2}}{\cot(\rho\pi) + \frac{1}{2} \frac{s^2}{(n-\mu)^2 - s^2} + \sqrt{(n-\mu)^2 - s^2}} + \right. \\ & \left. + \frac{i}{2} \exp \left[2\sqrt{(n-\mu)^2 - s^2} - 2(n-\mu) \operatorname{arccosh} \left(\frac{|n-\mu|}{s} \right) \right] \right\}^{-1}, \quad (\text{A.13}) \end{aligned}$$

where (41) is taken into account. The sum in (A.13) at $\varphi = 0$ is evaluated asymptotically at $s \gg 1$ by converting it into integral:

$$f_3(p, 0) = -\sqrt{\frac{2r_c}{\pi}} \left\{ \sqrt{s} \cos(\mu\pi) [I^{(1)}(s) - iI^{(2)}(s)] + O(1/\sqrt{s}) \right\}, \quad (\text{A.14})$$

where

$$I^{(1)}(s) = 4 \int_1^\infty dv \exp \left[2s \left(\text{varccosh} v - \sqrt{v^2 - 1} \right) \right] \frac{\cot(\rho\pi) + \frac{1}{2} \frac{1}{v^2 - 1} - s\sqrt{v^2 - 1}}{\cot(\rho\pi) + \frac{1}{2} \frac{1}{v^2 - 1} + s\sqrt{v^2 - 1}} \times \\ \times \left\{ 1 + 4 \exp \left[4s \left(\text{varccosh} v - \sqrt{v^2 - 1} \right) \right] \left[\frac{\cot(\rho\pi) + \frac{1}{2} \frac{1}{v^2 - 1} - s\sqrt{v^2 - 1}}{\cot(\rho\pi) + \frac{1}{2} \frac{1}{v^2 - 1} + s\sqrt{v^2 - 1}} \right]^2 \right\}^{-1} \quad (\text{A.15})$$

and

$$I^{(2)}(s) = 2 \int_1^\infty dv \times \\ \times \left\{ 1 + 4 \exp \left[4s \left(\text{varccosh} v - \sqrt{v^2 - 1} \right) \right] \left[\frac{\cot(\rho\pi) + \frac{1}{2} \frac{1}{v^2 - 1} - s\sqrt{v^2 - 1}}{\cot(\rho\pi) + \frac{1}{2} \frac{1}{v^2 - 1} + s\sqrt{v^2 - 1}} \right]^2 \right\}^{-1} \quad (\text{A.16})$$

In the cases of $|\cot(\rho\pi)| \gg s$ and $|\cot(\rho\pi)| \sim s$, the integrand in (A.15) is maximal at $v = 1$, and one can use the Laplace method for the estimation of $I^{(1)}(s)$ in this case, see, e.g., [29]. Namely, let $g(v)$ be continuous function with $0 < g(1) < \infty$ and $h(v)$ be continuous differentiable function obeying conditions $\frac{d}{dv}h(v) < 0$ at $v > 1$ and $\frac{d}{dv}h(v) \approx -a(v-1)^{\nu-1}$ at $v \rightarrow 1$ ($\nu > 0$), then

$$\int_1^\infty dv g(v) e^{sh(v)} = \frac{g(1)}{\nu} e^{sh(1)} \Gamma\left(\frac{1}{\nu}\right) \left(\frac{\nu}{as}\right)^{1/\nu}, \quad (\text{A.17})$$

where $\Gamma(y)$ is the Euler gamma-function. In the present case $g(1) = 1$, $h(1) = 0$, $a = 2\sqrt{2}$, $\nu = 3/2$, and we obtain

$$I^{(1)}(s) = \Gamma\left(\frac{2}{3}\right) (12s^2)^{-1/3}. \quad (\text{A.18})$$

A similar estimate is obtained for $I^{(2)}(s)$ (A.16) in the case of $|\cot(\rho\pi)| \gg s$:

$$I^{(2)}(s) = \frac{1}{4} \Gamma\left(\frac{2}{3}\right) (6s^2)^{-1/3}. \quad (\text{A.19})$$

In the case of $|\cot(\rho\pi)| \ll s$, the integrand in (A.16) is maximal at $v = v_0$ where $v_0 = 1 + 2^{-5/3}s^{-2/3}$. The integrand in the vicinity of the maximum is approximated as

$$2 \exp \left\{ -9(2s)^{4/3} \exp[10(2s)^{-2/3}](v - v_0)^2 \right\},$$

and the integral is estimated as

$$I^{(2)}(s) = \frac{\sqrt{\pi}}{3} \left(\frac{2}{s^2} \right)^{1/3}. \quad (\text{A.20})$$

The integrand in (A.15) in the case of $|\cot(\rho\pi)| \ll s$ changes sign at $v = v_0$. Similarly to (A.20), we obtain the estimate for integral $I^{(1)}(s)$ in this case:

$$I^{(1)}(s) = -\frac{1}{3} \left(\frac{1}{2s} \right)^{2/3}. \quad (\text{A.21})$$

In the case of $|\cot(\rho\pi)| \sim s$, the numerical analysis of $I^{(2)}(s)$ yields an estimate of order $O(s^{-2/3})$. Hence, we obtain the estimate for amplitude $f_3(p, \varphi)$ (A.13):

$$f_3(p, \varphi) \leq \sqrt{\frac{2r_c s}{\pi}} |I^{(1)}(s) - iI^{(2)}(s)| \leq \sqrt{r_c} a s^{-1/6}, \quad (\text{A.22})$$

where constant a is independent of s , μ and φ .

Appendix B. Hamilton-Jacobi equation of motion and classical cross section

In classical theory, we consider scattering of a point charged particle by an impenetrable tube containing magnetic flux Φ . The Hamilton function corresponding to the motion outside of the tube is

$$H = \frac{1}{2m} \left[p_r^2 + \frac{1}{r^2} (p_\theta - \mu')^2 + p_z^2 \right], \quad (\text{B.1})$$

where $\mu' = e\Phi(2\pi c)^{-1}$, and the Hamilton-Jacobi equation (see, e.g., [16]) is

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left[\left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \theta} - \mu' \right)^2 + \left(\frac{\partial S}{\partial z} \right)^2 \right] = 0. \quad (\text{B.2})$$

A general solution to (B.2) is

$$S = -Ht + \int_r^r dr p_r + \int_\theta^\theta d\theta p_\theta + \int_z^z dz p_z. \quad (\text{B.3})$$

Primary conservation laws are

$$p_\theta = \alpha, \quad p_z = \tilde{\alpha}, \quad \frac{1}{2m} \left[p_r^2 + \frac{1}{r^2} (p_\theta - \mu')^2 + p_z^2 \right] = E \quad (\text{B.4})$$

with α , $\tilde{\alpha}$ and E being arbitrary constants ($E > 0$). Taking the conservation laws into account, we obtain action S (B.3) in the form

$$S = -Et + \int_r^r dr \sqrt{2mE - \tilde{\alpha}^2 - r^{-2}(\alpha - \mu')^2} + \alpha\theta + \tilde{\alpha}z + \text{const.} \quad (\text{B.5})$$

Secondary conservation laws are

$$\frac{\partial S}{\partial \alpha} = \beta_\alpha, \quad \frac{\partial S}{\partial \tilde{\alpha}} = \beta_{\tilde{\alpha}}, \quad \frac{\partial S}{\partial E} = \beta_E \quad (\text{B.6})$$

with β_α , $\beta_{\tilde{\alpha}}$ and β_E being arbitrary constants. Thus we obtain relations

$$\beta_\alpha = \theta - \int_r^r dr \frac{r^{-2}(\alpha - \mu')}{\sqrt{2mE - \tilde{\alpha}^2 - r^{-2}(\alpha - \mu')^2}}, \quad (\text{B.7})$$

$$\beta_{\tilde{\alpha}} = z - \int_r^r dr \frac{\tilde{\alpha}}{\sqrt{2mE - \tilde{\alpha}^2 - r^{-2}(\alpha - \mu')^2}}, \quad (\text{B.8})$$

$$\beta_E = -t + \int_r^r dr \frac{m}{\sqrt{2mE - \tilde{\alpha}^2 - r^{-2}(\alpha - \mu')^2}}, \quad (\text{B.9})$$

which determine a trajectory of the particle, i.e. the functional dependence of θ , z and t on r . One can conclude that the classical trajectory is independent of the enclosed magnetic flux, since μ' can be absorbed into constant α : $\alpha - \mu' \rightarrow \alpha$. The trajectory is symmetric with respect to the point of reflection, $r = r_c$.

As a consequence of (B.7), we obtain the following expression for the incidence angle which is equal to the reflection angle:

$$\theta_{r_c} - \theta_\infty = \int_\infty^{r_c} dr \frac{r^{-2}\alpha}{\sqrt{p^2 - r^{-2}\alpha^2}} = -\arcsin\left(\frac{\alpha}{pr_c}\right), \quad (\text{B.10})$$

where $p = \sqrt{2mE - \tilde{\alpha}^2}$ ($2mE > \tilde{\alpha}^2$). Defining the impact parameter as

$$b = -\alpha p^{-1} \quad (-r_c < b < r_c) \quad (\text{B.11})$$

and using elementary tools of plane geometry, we determine the angle of deflection (called also the scattering angle) in range $-\pi < \varphi < \pi$ as

$$\varphi = \text{sgn}(b)\pi - 2(\theta_{r_c} - \theta_\infty) = 2\text{sgn}(b) \arccos(|b|/r_c). \quad (\text{B.12})$$

Hence, the scattering differential cross section is

$$\frac{d\sigma^{\text{class}}}{dzd\varphi} \equiv -\frac{db}{d\varphi} = \frac{r_c}{2} \left| \sin \frac{\varphi}{2} \right|. \quad (\text{B.13})$$

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